

Spin quantum correlations of relativistic particles

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We show that a pair of massive relativistic spin-1/2 particles prepared in a maximally entangled spin state in general is not capable of maximally violating the Clauser-Horne-Shimony-Holt (CHSH) version of Bell's inequalities without a post-selection of the particles momenta, representing a major difference in relation to non-relativistic systems. This occurs because the quantization axis of the measurements performed on each particle depends on the particle velocity, such that it is not possible to define a reduced density matrix for the particles spin. We also show that the amount of violation of the CHSH inequality depends on the reference frame, and that in some frames the inequality may not be violated.

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Special relativity and quantum mechanics were two theories constructed in the last century that completely changed our way to see nature, being the foundations of present-day theoretical physics. One of the most striking features of quantum mechanics is entanglement, that leads to quantum correlations among parties of a system that are stronger than what is allowed by classical physics [1–3]. Recently questions regarding the behaviour of the entropy and entanglement of quantum systems in different reference frames gave rise to the field of relativistic quantum information [4–6]. Since then, many studies of relativistic effects on spin quantum correlations of massive particles have appeared in the literature [4, 7–20].

In the seminal work of Peres, Scudo and Terno [5], they showed that the reduced density matrix for the spin of a relativistic particle, that should give “the statistical predictions for the results of measurements of spin components by an ideal apparatus which is not affected by the momentum of the particle” [5], is not covariant under Lorentz transformations. This occurs because under a Lorentz boost the particle spin undergoes a Wigner rotation [21, 22], which correspond to a momentum-dependent change of the particle spin state. One aspect extensively studied in the posterior works on relativistic quantum information was the influence of the Wigner rotations on the amount of entanglement of the reduced spin density matrix of a system with two or more particles in different reference frames [8, 10, 15–17, 19]. However, in our recent work [23] we discussed that since it is not possible to measure the spin of a relativistic particle in a independent way from its momentum, the definition of a reduced density matrix for the particle spin is meaningless. The ideal apparatus which is not affected by the particle momentum conjectured by Peres *et al.* [5] does not exist, contradicting the assumptions (explicitly or implicitly) made by the cited works [5, 6, 8, 10, 15–

17, 19, 20].

A second aspect studied in many of the previous works on the subject is the influence of the dependence of the Pauli-Lubanski (or similar) spin operators with the particles momenta on the amount of violation of Bell's inequalities with relativistic particles [4, 9, 12–14, 18, 19]. But in our recent work [23] we discussed that to use the Pauli-Lubanski (or similar) spin operators to describe spin measurements, spin must couple to a quantity that transforms as part of a 4-vector under Lorentz transformations in the measuring apparatus. But we do not know if such a coupling exists in nature. If spin couples to an electromagnetic field in the measuring apparatus, like in the Stern-Gerlach measurements, the spin operators must transform as part of a tensor under Lorentz transformations to guarantee the invariance of the outcomes probabilities, with different predictions in relation to the Pauli-Lubanski treatment [23].

For the reasons described in the previous paragraphs, we believe that the analysis of spin quantum correlations of relativistic systems must be revisited. Here we apply our method to the case of two entangled spin-1/2 massive particles to show how the maximum amount of violation of the Clauser-Horne-Shimony-Holt (CHSH) version of Bell's inequalities [3] depends on the velocity distribution of the particles. This represents a major difference in relation to non-relativistic systems, and was not considered in the previous works on the subject. We also show that observers in different reference frames may obtain different amounts of violation for the CHSH inequality, and that some of them may not be able to violate this inequality without a post-selection of the particles momenta.

In the present work, we will consider the following state for two spin-1/2 particles labelled by a and b in the center-of-mass rest frame:

$$|\Psi\rangle = \int d^3p \psi(\mathbf{p}) \left[|\mathbf{p}, \uparrow\rangle_a |\mathbf{p}, \downarrow\rangle_b - |\mathbf{p}, \downarrow\rangle_a |\mathbf{p}, \uparrow\rangle_b \right] \quad (1)$$

where $|\mathbf{p}, \uparrow\rangle$ ($|\mathbf{p}, \downarrow\rangle$) represents a state with momentum

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\mathbf{p} and spin pointing in the $\hat{\mathbf{z}}$ ($-\hat{\mathbf{z}}$) direction, with $\psi(\mathbf{p})$ given, in spherical coordinates (r, θ, ϕ) , by

$$\psi(p, \theta_p, \phi_p) \propto \delta(p - m_b \gamma_{v_b} v_b), \quad (2)$$

m_b being the mass of particle b , v_b the modulus of its velocity and $\gamma_{v_b} \equiv 1/\sqrt{1-v_b^2}$. We are using a system of units in which the speed of light in vacuum is $c = 1$. Considering the state of Eq. (1) with the wavefunction of Eq. (2), we see that each particle can go in any direction, but the particles propagate always in opposite directions and with the same absolute value for the momentum, in a singlet state of spin. This state can be obtained, for instance, through the decay of a spin-0 particle in two spin-1/2 particles in the center-of-mass rest frame of the system. The form of the state is a direct consequence of the conservation of momentum and angular momentum in the process. In this paper, we are following the Wigner's definition of spin [21], that correspond to the angular momentum of the particle in its own rest frame. We will also consider here that the mass of particle a is much greater than the mass of particle b , such that particle a has non-relativistic velocities in the state of Eq. (1). This last assumption is made only to maximize the relativistic effects that will be presented.

Note that if the spin and the momentum of relativistic particles could be treated as independent variables, the state of Eq. (1) could be written as the product of a state for the particles momenta and a state for the particles spin. Upon tracing out the particles momenta, we would have a maximally entangled state for the particles spin. However, as we will discuss in this paper, it is not possible to treat the spin and the momentum of relativistic particles as independent variables, so we cannot trace out the particles momenta and define a reduced density matrix for the particles spin.

If we make joint measurements of spin in both particles considering eigenvalues ± 1 for each measurement, the CHSH version of Bell's inequalities states that for any local realistic description of the correlations among the particles we must have [3]

$$S = |\langle \hat{\mathbf{a}}_1, \hat{\mathbf{b}}_1 \rangle + \langle \hat{\mathbf{a}}_1, \hat{\mathbf{b}}_2 \rangle + \langle \hat{\mathbf{a}}_2, \hat{\mathbf{b}}_1 \rangle - \langle \hat{\mathbf{a}}_2, \hat{\mathbf{b}}_2 \rangle| \leq 2, \quad (3)$$

where $\langle \hat{\mathbf{a}}_i, \hat{\mathbf{b}}_j \rangle \equiv \langle \mathbf{a}_i \cdot \hat{\boldsymbol{\sigma}} \otimes \mathbf{b}_j \cdot \hat{\boldsymbol{\sigma}} \rangle$ with \mathbf{a}_i and \mathbf{b}_j being unit vectors and $\hat{\boldsymbol{\sigma}} \equiv \hat{\sigma}_x \hat{\mathbf{x}} + \hat{\sigma}_y \hat{\mathbf{y}} + \hat{\sigma}_z \hat{\mathbf{z}}$, $\hat{\sigma}_x$, $\hat{\sigma}_y$ and $\hat{\sigma}_z$ being the spin-1/2 Pauli matrices. $\langle \hat{\mathbf{a}}_i, \hat{\mathbf{b}}_j \rangle$ represents the expectation value of a joint spin measurement with quantization axis in the direction \mathbf{a}_i for particle a and in the direction \mathbf{b}_j for particle b . For the state of Eq. (1) we have $\langle \hat{\mathbf{a}}_i, \hat{\mathbf{b}}_j \rangle = -\mathbf{a}_i \cdot \mathbf{b}_j$, such that for $\mathbf{a}_1 = \hat{\mathbf{x}}$, $\mathbf{a}_2 = \hat{\mathbf{y}}$, $\mathbf{b}_1 = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$ and $\mathbf{b}_2 = (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$ we have $S = 2\sqrt{2} \approx 2.83 > 2$, indicating that quantum mechanics is not a local realistic theory. This fact makes entangled states like the one of Eq. (1) very important in the field of quantum information.

However, considering that particle b has relativistic velocities \mathbf{v}_b in the state of Eq. (1), if we physically imple-

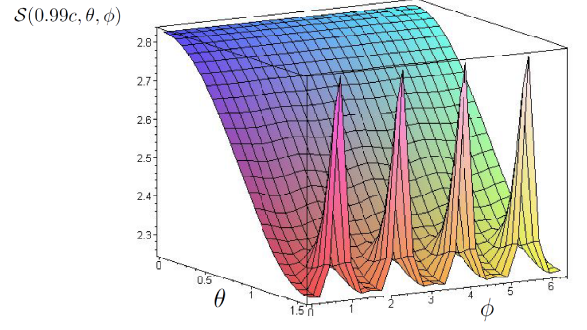


FIG. 1: $\mathcal{S}(v_b, \theta, \phi)$, defined as S from Eq. (3) for spin measurements made by Stern-Gerlach apparatuses with magnetic fields in the directions $\mathbf{B}_{a1} \propto \hat{\mathbf{x}}$, $\mathbf{B}_{a2} \propto \hat{\mathbf{y}}$, $\mathbf{B}_{b1} \propto (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$ and $\mathbf{B}_{b2} \propto (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$ for the state of Eq. (1) with the wavefunction of Eq. (2) with $v_b = 0.99c$ in function of the direction of the velocity of particle b in spherical coordinates.

ment a spin measurement using an experimental apparatus of the Stern-Gerlach type, in which spin couples to an electromagnetic field, each velocity component of the particle will see a different quantization axis for the measurement. This occurs because, since spin refers to the particle angular momentum in its own rest frame, being proportional to the particle magnetic moment in the rest frame, the quantization axis of a spin measurement is in the direction of the apparatus magnetic field in the particle rest frame [23]. But to compute the magnetic field in the particle rest frame from the magnetic field in the laboratory frame we must apply the corresponding Lorentz transformation, that depends on the particle velocity. If the apparatus magnetic field is \mathbf{B} in the laboratory frame (assumed here to be the same as the particles center-of-mass rest frame), in the particle b rest frame it will be [24]

$$\mathbf{B}_0 = \gamma_{v_b} \mathbf{B} - \frac{\gamma_{v_b}^2 (\mathbf{v}_b \cdot \mathbf{B})}{\gamma_{v_b} + 1} \mathbf{v}_b, \quad (4)$$

with $\gamma_{v_b} \equiv 1/\sqrt{1-v_b^2}$, and the quantization axis of the measurement will be in the direction of \mathbf{B}_0 . If we consider CHSH measurements with the magnetic fields of the apparatuses in the directions $\mathbf{B}_{a1} \propto \hat{\mathbf{x}}$, $\mathbf{B}_{a2} \propto \hat{\mathbf{y}}$, $\mathbf{B}_{b1} \propto (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$ and $\mathbf{B}_{b2} \propto (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$ in the laboratory frame, the amount of violation of the inequality of Eq. (3) will depend on \mathbf{v}_b , since the quantization axes of the measurements depend on \mathbf{v}_b , and we can define a function $\mathcal{S}(v_b, \theta, \phi)$ that includes the dependence of the quantity S from Eq. (3) on the velocity of particle b in spherical coordinates. Fig. 1 illustrates $\mathcal{S}(v_b, \theta, \phi)$ for $v_b = 0.99c$ and v_a non-relativistic. We see that the amount of violation of the Bell's inequality depend on the direction of propagation of the particle. However, for each particle velocity we can choose other pair of directions for the magnetic fields \mathbf{B}_{b1} and \mathbf{B}_{b2} in the laboratory frame such that the Bell's inequality is maximally violated. But if we are making spin measurements

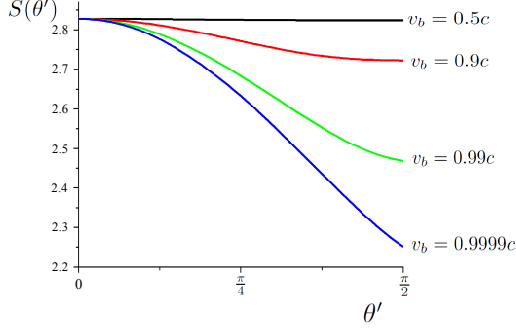


FIG. 2: $S(\theta')$ from Eq. (5) for spin measurements made by Stern-Gerlach apparatuses with magnetic fields in the directions $\mathbf{B}_{a1} \propto \hat{\mathbf{x}}$, $\mathbf{B}_{a2} \propto \hat{\mathbf{y}}$, $\mathbf{B}_{b1} \propto (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$ and $\mathbf{B}_{b2} \propto (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$ for the state of Eq. (1) with the wavefunction of Eq. (2) with $v_b = 0.5c$, $v_b = 0.9c$, $v_b = 0.99c$ and $v_b = 0.9999c$.

with detectors that have an acceptance angle θ' , making a post-selection of particles b propagating in directions within cones with $\theta < \theta'$, an average of the situations illustrated in the graph of Fig. 1 will be the result of the measurement of the quantity S from Eq. (3):

$$S(\theta') = \frac{\int_0^{\theta'} d\theta \int_0^{2\pi} d\phi \sin(\theta) S(v_b, \theta, \phi)}{\int_0^{\theta'} d\theta \int_0^{2\pi} d\phi \sin(\theta)}, \quad (5)$$

since each momentum component that enters in the detector in general will have different quantization axes for the spin measurements. Fig. 2 shows the values of the quantity $S(\theta')$ from Eq. (5) for different modulus of the velocities v_b in the wavefunction of Eq. (2). The smaller is the acceptance angle of the detector θ' , higher is the value of $S(\theta')$, but smaller is the fraction of particles that are detected.

Fig. 2 shows that the amount of violation of Bell's inequalities for the state of Eq. (1) depends on the velocity distribution of the particles, even the reduced spin density matrix of this state being maximally entangled. Only with a post-selection of particles b in momentum eigenstates ($\theta' \rightarrow 0$ in Fig. 2) we obtain a maximal violation. This example illustrates why the association of a reduced spin density matrix for relativistic particles and the quantification of the entanglement of this reduced state, as frequently done in the literature [8, 10, 15–17, 19, 20], is meaningless. We cannot predict the expectation values of measurements without considering the velocities of the particles. It may be argued that this fact occurs because we choose Stern-Gerlach apparatuses to implement the measurements, and that it may exist other kinds of apparatuses that measure the particle spin independently of the particle velocity, as conjectured by Peres *et al.* [5]. But a spin measurement must be made through the coupling of the 3 components of spin with some 3-component quantity of the measuring apparatus, represented by a

scalar interaction Hamiltonian. In a covariant treatment, the interaction Hamiltonian must be proportional to an invariant scalar to guarantee that all observers in inertial reference frames compute the same expectation values for the spin measurements [23]. This fact implies that spin must transform under Lorentz transformation in the same way as the physical quantity to which it couples in the measuring apparatus. So the quantity that couples with spin in the measuring apparatus computed in the particle rest frame will, in any case, depend on the particle velocity, such that it is not possible to measure the particle spin independently from its velocity (or momentum), no matter how we measure spin.

If the particles center-of-mass propagates with velocity $\beta = \beta \hat{\mathbf{z}}$ in relation to the laboratory frame, the state viewed in the laboratory frame would be the state of Eq. (1) transformed by a Lorentz boost with velocity $-\beta$. However, to see what would be the amount of violation of the Bell's inequality of Eq. (3) in this situation, it is easier to compute the direction of the apparatuses magnetic fields in each particle rest frame. Since the directions of the magnetic fields of the Stern-Gerlach apparatuses in the laboratory frame to be used are orthogonal to the center-of-mass velocity, in the center-of-mass rest frame the electromagnetic fields are [24] $\mathbf{B}' = \gamma_\beta \mathbf{B}$ and $\mathbf{E}' = \gamma_\beta \boldsymbol{\beta} \times \mathbf{B}$ with $\gamma_\beta \equiv 1/\sqrt{1 - \beta^2}$, \mathbf{B} being the magnetic field in the laboratory frame and \mathbf{B}' and \mathbf{E}' being the magnetic and electric fields in the particles center-of-mass frame. Since particle a is assumed to have a nonrelativistic velocity in relation to the particles center-of-mass, the quantization axis of each measurement for this particle is in the same direction as the magnetic field in the laboratory frame. However, since particle b is assumed to have relativistic velocities \mathbf{v}_b in relation to the particles center-of-mass, the quantization axis for each velocity component is in the direction of the vector [24]

$$\mathbf{B}'_0 = \gamma_{v_b} \gamma_\beta [\mathbf{B} - \mathbf{v}_b \times (\boldsymbol{\beta} \times \mathbf{B})] - \frac{\gamma_\beta \gamma_{v_b}^2}{\gamma_{v_b} + 1} \mathbf{v}_b (\mathbf{v}_b \cdot \mathbf{B}), \quad (6)$$

that represents the magnetic field in the particle rest frame.

We can compute the amount of violation of Bell's inequality in Eq. (3) considering the above dependence of the quantization axis of the spin measurement of particle b with \mathbf{v}_b , $\boldsymbol{\beta}$ and \mathbf{B} . Choosing again $\mathbf{B}_{a1} \propto \hat{\mathbf{x}}$, $\mathbf{B}_{a2} \propto \hat{\mathbf{y}}$, $\mathbf{B}_{b1} \propto (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$ and $\mathbf{B}_{b2} \propto (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$, considering the state from Eq. (1) in the particles center-of-mass frame with the wavefunction of Eq. (2) with $v_b = 0.99c$, we plot in Fig. 3 the values for $S(\theta')$ from Eq. (5), noting that now S also depends on $\boldsymbol{\beta}$, for different velocities $\boldsymbol{\beta}$ between the particles center-of-mass and the laboratory frame. Note that a post-selection of particles b propagating in directions within cones with $\theta < \theta'$ in relation to the particles center-of-mass correspond to different acceptance angles θ'' of the detectors for each velocity $\boldsymbol{\beta}$. We opted for this representation in the graphics of Fig. 3 because in this way we are always post-selecting the same portion of the state in each situation. We can see

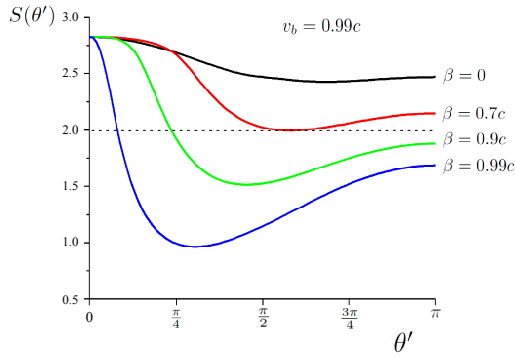


FIG. 3: $S(\theta')$ from Eq. (5) for spin measurements made by Stern-Gerlach apparatuses with magnetic fields in the directions $\mathbf{B}_{a1} \propto \hat{\mathbf{x}}$, $\mathbf{B}_{a2} \propto \hat{\mathbf{y}}$, $\mathbf{B}_{b1} \propto (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$ and $\mathbf{B}_{b2} \propto (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$ for the state of Eq. (1) with the wavefunction of Eq. (2) with $v_b = 0.99c$ in a reference frame that moves with velocity $\beta\hat{\mathbf{z}}$ in relation to the center-of-mass rest frame, with $\beta = 0$, $\beta = 0.7c$, $\beta = 0.9c$ and $\beta = 0.99c$.

that the amount of violation of Bell's inequality is different in different reference frames, and that in some frames we cannot even violate it without a post-selection of the state.

Let us consider now another situation in which Alice prepares the state of Eq. (1) in her laboratory, with the particles rest frame coinciding with the laboratory frame, and send particle b with a selected velocity \mathbf{v}_b to Bob. Consider that Alice's reference frame is moving with a relativistic velocity $\beta = \beta\hat{\mathbf{z}}$ in relation to Bob's frame. If they want to maximally violate a Bell's inequality with a

set of shared pair of particles obtained in this way, Bob has to adjust his Stern-Gerlach apparatus with magnetic fields in directions \mathbf{B} such that the directions of the vectors \mathbf{B}'_0 from Eq. (6) are the directions that, together with the quantization axes chosen by Alice, maximally violate Eq. (3).

In summary, we had shown how the dependence of the quantization axis of a spin measurement on the particle velocity influences the violation of the CHSH version of Bell's inequalities with relativistic spin-1/2 particles in different situations. Our treatment takes into account how spin measurements can be physically implemented, making different predictions in relation to the previous works on the subject. Even if the particles are in a maximally entangled spin state, we may not maximally violate the Bell's inequalities without a post-selection of the particles momenta, such that the definition of a reduced density matrix for the particles spin is meaningless. The spin and the momentum of relativistic particles cannot be considered independent variables in a relativistic treatment. We also showed that observers in different reference frames measure different amounts of violation for the CHSH inequality, and that in some reference frames the inequality may not be violated. We believe that many of our predictions can be experimentally tested with an apparatus similar to the one from Sakai *et al.* experiments [25].

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